Complementary note on:

Convergence analysis for error e_1 of the nonholonomic manipulator in the VFO set-point control system

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January 18, 2014

This note complements paper [1] presenting the detailed convergence analysis for error e_1. References to equations taken from original paper [1] will be denoted in brackets '$\{\cdot\}$'. Section 1 presents convergence analysis for error e_1 in the case where errors e_2 and e_3 tend toward zero in infinite time (the *infinite-time case*). Comments on the convergence of e_1 in the case of finite-time convergence of e_2 and e_3 (the *finite-time case*) are provided in Section 2.

1 Convergence analysis for error e_1 – infinite-time (IT) case

For IT case we have $\rho \equiv 1$, thus equation $\{43\}$ for $e_a \to 0$ takes the form

$$\dot{\boldsymbol{e}}^* = s \cdot (-k_p \boldsymbol{e}^* - \boldsymbol{v}^*), \quad \text{where} \quad \boldsymbol{e}^* = [e_2 \; e_3]^T \tag{1}$$

or in the component-wise form (using $\{29\}$ and $\{30\}$):

$$\dot{e}_2 = s \cdot \left(-k_p e_2 + \eta\sigma \, \| \boldsymbol{e}^* \| \, \frac{\sin q_{1t}}{\| \boldsymbol{g}_{2t}^* \|} \right), \tag{2}$$

$$\dot{e}_3 = s \cdot \left(-k_p e_3 + \eta\sigma \, \| \boldsymbol{e}^* \| \, \frac{\cos q_{1t} \cos q_{2t}}{\| \boldsymbol{g}_{2t}^* \|} \right). \tag{3}$$

The subsequent analysis will be presented in the three main steps reflecting our reasoning.

1.1 Step1: transformation of error $\boldsymbol{e}^*$ and its dynamics to the new space

Let us conduct the analysis in the auxiliary error space using the following error transformation:

$$\boldsymbol{e}_L^* = \begin{bmatrix} e_{2L} \\ e_{3L} \end{bmatrix} \triangleq \boldsymbol{L}\boldsymbol{e}^* = \begin{bmatrix} \sin q_{1t} & \cos q_{1t} \cos q_{2t} \\ -\cos q_{1t} \cos q_{2t} & \sin q_{1t} \end{bmatrix} \begin{bmatrix} e_2 \\ e_3 \end{bmatrix}, \tag{4}$$

where $\boldsymbol{e}_L^*$ is the transformed error vector, and $\boldsymbol{L}$ is the transformation matrix. Note that

$$\boldsymbol{L} = \begin{bmatrix} \sin q_{1t} & \cos q_{1t} \cos q_{2t} \\ -\cos q_{1t} \cos q_{2t} & \sin q_{1t} \end{bmatrix} \quad \Rightarrow \quad \det(\boldsymbol{L}) = \| \boldsymbol{g}_{2t}^* \|^2 \neq 0, \tag{5}$$

where the last relation results from definition $\{10\}$ (matrix $\boldsymbol{L}$ is invertible). Due to the above relation it is clear that the convergence $\boldsymbol{e}^* \to \boldsymbol{0}$ (it has been proved in [1]) implies $\boldsymbol{e}_L^* \to \boldsymbol{0}$.

Now, by time-differentiating equation (4) and using (2)-(3) one can obtain the following dynamics of the transformed error components:

$$\dot{e}_{2L} = s \cdot (-k_p e_{2L} + \eta\sigma \, \| \boldsymbol{e}_L^* \|), \tag{6}$$

$$\dot{e}_{3L} = s \cdot (-k_p e_{3L}), \tag{7}$$

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where in (6) we have used the fact that (see Appendix 3.1)

$$\|\boldsymbol{e}^*\| = \frac{\|\boldsymbol{e}_L^*\|}{\|\boldsymbol{g}_{2t}^*\|}, \qquad \|\boldsymbol{e}_L^*\| = \sqrt{e_{2L}^2 + e_{3L}^2}. \tag{8}$$

Equations (6)-(7) represent dynamics of error $\boldsymbol{e}^*$ in the new space.

Let us define the decision factor as follows:

$$\sigma \triangleq \operatorname{sgn}(e_{2L0}) \overset{(4)}{=} \operatorname{sgn}(e_2(0)\sin q_{1t} + e_3(0)\cos q_{1t}\cos q_{2t}) \in \{-1, +1\}, \tag{9}$$

where $e_{2L0} \equiv e_{2L}(\tau = 0)$ is the initial value of the error component e_{2L}. According to (4)-(5), due to convergence $\boldsymbol{e}^* \to \boldsymbol{0}$ (proved in [1]), and due to the form of (6)-(7) along with (9) allow us to formulate the following corollaries:

C1) the error components $e_{2L}(\tau)$ and $e_{3L}(\tau)$ converge asymptotically to zero,

C2) $e_{3L}(\tau)$ converges to zero independently of $e_{2L}(\tau)$ and it terminally passes $e_{2L}(\tau)$,

C3) using (9) in (6) gives: $\dot{e}_{2L} = s \cdot (-k_p e_{2L} + \eta \operatorname{sgn}(e_{2L0}) \|\boldsymbol{e}_L^*\|)$; hence (since $\eta > 0$) the solution $e_{2L}(\tau)$ preserves its initial sign, namely:

$$\forall_{\tau \geq 0} \ \operatorname{sgn}(e_{2L}(\tau)) = \operatorname{sgn}(e_{2L0}). \tag{10}$$

1.2 Step2: transformation of $\boldsymbol{h}^*$ and convergence of its argument in the new space

Similarly as in (4) let us transform the convergence vector field $\boldsymbol{h}^* = [h_2 \ h_3]^T$ as follows:

$$\boldsymbol{h}_L^* = \begin{bmatrix} h_{2L} \\ h_{3L} \end{bmatrix} \triangleq \boldsymbol{L}\boldsymbol{h}^* = \begin{bmatrix} \sin q_{1t} & \cos q_{1t}\cos q_{2t} \\ -\cos q_{1t}\cos q_{2t} & \sin q_{1t} \end{bmatrix} \begin{bmatrix} h_2 \\ h_3 \end{bmatrix}, \tag{11}$$

which after using {27} and {29} can be rewritten as

$$\boldsymbol{h}_L^* \triangleq \boldsymbol{L}\boldsymbol{h}^* = k_p \boldsymbol{L}\boldsymbol{e}^* - \eta\sigma \frac{\|\boldsymbol{e}^*\|}{\|\boldsymbol{g}_{2t}^*\|} \boldsymbol{L}\boldsymbol{g}_{2t}^* = k_p \boldsymbol{e}_L^* - \eta\sigma \|\boldsymbol{e}^*\| \begin{bmatrix} \|\boldsymbol{g}_{2t}^*\| \\ 0 \end{bmatrix}, \tag{12}$$

since $\boldsymbol{L}\boldsymbol{g}_{2t}^* = [\|\boldsymbol{g}_{2t}^*\|^2 \ \ 0]^T$ (by direct computations). Equation (12) has the following component-wise form:

$$\boldsymbol{h}_L^* = \begin{bmatrix} h_{2L} \\ h_{3L} \end{bmatrix} = \begin{bmatrix} k_p e_{2L} - \eta\sigma \|\boldsymbol{e}_L^*\| \\ k_p e_{3L} \end{bmatrix} = \frac{1}{s} \begin{bmatrix} -\dot{e}_{2L} \\ -\dot{e}_{3L} \end{bmatrix}, \tag{13}$$

where the last equality results from comparison with (6)-(7). Now, defining the argument

$$\beta \triangleq \operatorname{Arg}(\boldsymbol{h}_L^*) \triangleq \operatorname{Atan2}(h_{3L}, h_{2L})$$

one can write:

$$\tan\beta(\tau) = \frac{h_{3L}(e_{3L}(\tau))}{h_{2L}(e_{2L}(\tau), e_{3L}(\tau))} \overset{(13)}{=} \frac{k_p e_{3L}(\tau)}{k_p e_{2L}(\tau) - \eta\sigma\sqrt{e_{2L}^2(\tau) + e_{3L}^2(\tau)}}. \tag{14}$$

Thus, due to corollaries C1) and C2) we obtain:

$$\tan\beta(\tau) \to 0 \quad \text{as} \quad \tau \to \infty. \tag{15}$$

The above convergence property is crucial for the subsequent analysis conducted in the next subsection.

1.3 Step3: relation between convergence of $\tan\beta$ and convergence of error e_1

In order to show relation between convergence of $\tan\beta$ to zero and convergence of error e_1 let us first rewrite the term $\tan\beta$ using definition (11):

$$\tan\beta = \frac{h_{3L}}{h_{2L}} \overset{(11)}{=} \frac{h_3 \sin q_{1t} - h_2 \cos q_{1t}\cos q_{2t}}{h_3 \cos q_{1t}\cos q_{2t} + h_2 \sin q_{1t}} = \frac{\left(\tan q_{1t} \frac{1}{\cos q_{2t}} - \frac{h_2}{h_3}\right) h_3 \cos q_{1t}\cos q_{2t}}{\left(1 + \tan q_{1t} \frac{1}{\cos q_{2t}} \frac{h_2}{h_3}\right) h_3 \cos q_{1t}\cos q_{2t}} =$$

$$= \frac{\left(\tan q_{1t} \frac{1}{\cos q_{2t}} - \tan q_{1a} \frac{1}{\cos q_2}\right)}{\left(1 + \tan q_{1t} \frac{1}{\cos q_{2t}} \tan q_{1a} \frac{1}{\cos q_2}\right)} = \frac{\tan\delta_t - \tan\delta_a}{1 + \tan\delta_t \tan\delta_a} = \tan(\delta_t - \delta_a), \tag{16}$$

where we have used the fact that $\tan q_{1a} = h_2 \cos q_2 / h_3$ (see {23}). Now, we can recall (15) which according to (16) implies that:

$$\tan\beta(\tau) \to 0 \quad \text{as} \quad \tau \to \infty \qquad \Rightarrow \qquad (\tan\delta_t - \tan\delta_a(\tau)) \to 0 \quad \text{as} \quad \tau \to \infty,$$

and as a consequence of (16)

$$\left(\tan q_{1t} \frac{1}{\cos q_{2t}} - \tan q_{1a}(\tau) \frac{1}{\cos q_2(\tau)}\right) \to 0 \quad \text{as} \quad \tau \to \infty. \tag{17}$$

Since $e_2(\tau) = q_{2t} - q_2(\tau) \to 0$ as $\tau \to \infty$ (see [1]) and $\cos q_2(\tau), \cos q_{2t} > 0$ for all $\tau \geq 0$ and all admissible q_{2t}, the convergence result (17) is equivalent to

$$(\tan q_{1t} - \tan q_{1a}(\tau)) \to 0 \quad \text{as} \quad \tau \to \infty. \tag{18}$$

Since $e_a(\tau) = q_{1a}(\tau) - q_1(\tau)$ asymptotically tends to zero (compare {42}), then finally (18) can be replaced with:

$$(\tan q_{1t} - \tan q_1(\tau)) \to 0 \quad \text{as} \quad \tau \to \infty. \tag{19}$$

Obviously, (19) can be met in two cases: when $q_1(\tau) \to q_{1t}$ or when $q_1(\tau) \to (q_{1t} \pm \pi)$. Note that:

$$\boldsymbol{g}_{2t}^{*T} \boldsymbol{g}_2^*(q_1 = q_{1t}, q_2 = q_{2t}) = + \| \boldsymbol{g}_{2t}^* \|^2 > 0, \tag{20}$$

$$\boldsymbol{g}_{2t}^{*T} \boldsymbol{g}_2^*(q_1 = q_{1t} \pm \pi, q_2 = q_{2t}) = - \| \boldsymbol{g}_{2t}^* \|^2 < 0. \tag{21}$$

Hence, to conclude that (19) holds along with (20), it suffices to show that $\operatorname{sgn}(\boldsymbol{g}_{2t}^{*T} \boldsymbol{g}_2^*(q_1, q_2)) = +1$ as $e_a, e_2, e_3 \to 0$ or equivalently as $e_a, e_{2L}, e_{3L} \to 0$. Because $\boldsymbol{g}_2^*(q_1, q_2) = \dot{\boldsymbol{q}}^*/U_2 = [\dot{q}_2/U_2 \ \dot{q}_3/U_2]^T$ (see {15}) one can write:

$$\operatorname{sgn}(\boldsymbol{g}_{2t}^{*T} \boldsymbol{g}_2^*(q_1, q_2)) = \operatorname{sgn}(U_2) \operatorname{sgn}(\dot{q}_2(q_1) \sin q_{1t} + \dot{q}_3(q_1, q_2) \cos q_{1t} \cos q_{2t}).$$

Furthermore, for $e_{1a} = 0$ we have $\sigma h_2 = b\dot{q}_2 \operatorname{sgn}(U_2)$ and $\sigma h_3 = b\dot{q}_3 \operatorname{sgn}(U_2)$ where $b > 0$ is some positive constant (compare {23} with $q_1 = \operatorname{Atan2c}(\operatorname{sgn}(U_2)\dot{q}_2 \cos q_2, \operatorname{sgn}(U_2)\dot{q}_3)$ computed according to {15}), thus we can write:

$$\begin{aligned}
\lim_{e_{1a}\to 0} \operatorname{sgn}(\boldsymbol{g}_{2t}^{*T} \boldsymbol{g}_2^*(q_{1a} - e_{1a}, q_2)) &= \lim_{e_a \to 0} \operatorname{sgn}(U_2) \operatorname{sgn}(\dot{q}_2(q_{1a} - e_{1a}) \sin q_{1t} + \dot{q}_3(q_{1a} - e_{1a}, q_2) \cos q_{1t} \cos q_{2t}) = \\
&= \operatorname{sgn}(U_2) \operatorname{sgn}\left(\frac{\sigma h_2}{b \operatorname{sgn}(U_2)} \sin q_{1t} + \frac{\sigma h_3}{b \operatorname{sgn}(U_2)} \cos q_{1t} \cos q_{2t}\right) = \\
&= \operatorname{sgn}(b) \operatorname{sgn}(\sigma h_2 \sin q_{1t} + \sigma h_3 \cos q_{1t} \cos q_{2t}) = \\
&= \sigma \operatorname{sgn}(h_2 \sin q_{1t} + h_3 \cos q_{1t} \cos q_{2t}) = \\
&\overset{(11)}{=} \sigma \operatorname{sgn}(h_{2L}) \overset{(12)}{=} \sigma \operatorname{sgn}(k_p e_{2L} - \eta\sigma\sqrt{e_{2L}^2 + e_{3L}^2}),
\end{aligned} \tag{22}$$

where $\boldsymbol{g}_2^*(q_1, q_2) = \boldsymbol{g}_2^*(q_{1a} - e_{1a}, q_2)$ since $e_{1a} \triangleq q_{1a} - q_1$. Because $e_{3L}(\tau)$ terminally passes $e_{2L}(\tau)$ (see C2)) we can write:

$$\begin{aligned}
\lim_{e_{2L}, e_{3L} \to 0} \left(\lim_{e_{1a}\to 0} \operatorname{sgn}(\boldsymbol{g}_{2t}^{*T} \boldsymbol{g}_2^*(e_{1a}, \cdot))\right) &\overset{(22)}{=} \lim_{e_{2L}\to 0} \lim_{e_{3L}\to 0} \sigma \operatorname{sgn}(k_p e_{2L} - \eta\sigma\sqrt{e_{2L}^2 + e_{3L}^2}) = \\
&= \lim_{e_{2L}\to 0} \sigma \operatorname{sgn}(k_p e_{2L} - \eta\sigma\, |e_{2L}|) = \\
&\overset{(9)}{=} \lim_{e_{2L}\to 0} \operatorname{sgn}(e_{2L0}) \operatorname{sgn}(k_p e_{2L} - \eta \operatorname{sgn}(e_{2L0})\, |e_{2L}|) = \\
&\overset{(10)}{=} \lim_{e_{2L}\to 0} \operatorname{sgn}(e_{2L0}) \operatorname{sgn}(k_p e_{2L} - \eta \operatorname{sgn}(e_{2L})\, |e_{2L}|) = \\
&= \lim_{e_{2L}\to 0} \operatorname{sgn}(e_{2L0}) \operatorname{sgn}(e_{2L}) \operatorname{sgn}(k_p - \eta) = \\
&\overset{(10)}{=} \lim_{e_{2L}\to 0} \operatorname{sgn}^2(e_{2L0}) \operatorname{sgn}(k_p - \eta) = 1,
\end{aligned} \tag{23}$$

where in the last stage we have used the fact that $\eta \in (0, k_p)$ from assumption (see {29}).

Combination of the results (19) and (23) allows concluding about terminal convergence of error $e_1(\tau) = f_1(q_{1t} - q_1(\tau)) \in (-\pi, \pi]$ (see {9}) in the sense:

$$e_1(\tau) \to 0 \quad \text{as} \quad \tau \to \infty. \tag{24}$$

2 Convergence analysis for error e_1 – finite-time (FT) case

For FT case equation {43} takes the following form for $e_a \to 0$:

$$\dot{\boldsymbol{e}}^* = \rho\, s\,(-k_p \boldsymbol{e}^* - \boldsymbol{v}^*), \quad \text{where} \quad \boldsymbol{e}^* = [e_2\ e_3]^T \tag{25}$$

or in the component-wise form (using {29} and {30}):

$$\dot{e}_2 = \rho\, s \left(-k_p e_2 + \eta\sigma \,\|\, \boldsymbol{e}^* \| \frac{\sin q_{1t}}{\|\, \boldsymbol{g}^*_{2t} \|}\right), \tag{26}$$

$$\dot{e}_3 = \rho\, s \left(-k_p e_3 + \eta\sigma \,\|\, \boldsymbol{e}^* \| \frac{\cos q_{1t} \cos q_{2t}}{\|\, \boldsymbol{g}^*_{2t} \|}\right), \tag{27}$$

where $\rho = \rho(\|\, \boldsymbol{e}^* \|)$ and $s \in (0, 1]$ are the strictly positive functions (see {52} and {35}).

The subsequent analysis will be presented in the three main steps reflecting our reasoning.

2.1 Step1$_{\mathbf{FT}}$: transformation of error $\boldsymbol{e}^*$ and its dynamics in the new space

Let us conduct the analysis in the auxiliary error space using the following error transformation (4). By time-differentiation of (4) and using (26)-(27) one can obtain the following dynamics of the transformed error components:

$$\dot{e}_{2L} = \rho\, s\,(-k_p e_{2L} + \eta\sigma \,\|\, \boldsymbol{e}^*_L \|), \tag{28}$$

$$\dot{e}_{3L} = \rho\, s\,(-k_p e_{3L}), \tag{29}$$

where we have used (8) (see Appendix 3.1). Equations (28)-(29) represent dynamics of error $\boldsymbol{e}^*$ in the new space for the FT case. Defining the decision factor as in (9), and according to (28)-(29) allow us to formulate the following corollaries:

C1$_{\text{FT}}$) the error components $e_{2L}(\tau)$ and $e_{3L}(\tau)$ converge to zero in finite time,

C2$_{\text{FT}}$) $e_{3L}(\tau)$ converges to zero independently of $e_{2L}(\tau)$ and it terminally passes $e_{2L}(\tau)$, namely:

$$\lim_{\tau \to \tau_2} e_{2L}(\tau) = 0, \quad \lim_{\tau \to \tau_3} e_{3L}(\tau) = 0, \quad \tau_2 > \tau_3 \tag{30}$$

C3$_{\text{FT}}$) using (9) in (6) gives: $\dot{e}_{2L} = \rho\, s\,(-k_p e_{2L} + \eta\, \text{sgn}(e_{2L0}) \,\|\, \boldsymbol{e}^*_L \|)$; hence (since $\eta > 0$) the solution $e_{2L}(\tau)$ preserves its initial sign, namely:

$$\forall_{\tau \geq 0}\ \text{sgn}(e_{2L}(\tau)) = \text{sgn}(e_{2L0}). \tag{31}$$

2.2 Step2$_{\mathbf{FT}}$: transformation of $\boldsymbol{h}^*$ and convergence of its argument in the new space

Since almost all considerations included in Subsection 1.2 do not depend on the convergence type of position errors, they remain valid also in FT case. Only the last conclusion formulated in (15) now holds in finite time, and according to (14) and (30) we obtain:

$$\tan\beta(\tau) \to 0 \quad \text{as} \quad \tau \to \tau_3, \qquad \tau_3 < \infty. \tag{32}$$

2.3 Step3$_{\mathbf{FT}}$: relation between convergence of $\tan\beta$ and convergence of error e_1

Recalling (16), (17), and (32) one concludes that for FT case holds:

$$\left(\tan q_{1t} \frac{1}{\cos q_{2t}} - \tan q_{1a}(\tau) \frac{1}{\cos q_2(\tau)}\right) \to 0 \quad \text{as} \quad \tau \to \tau_3, \qquad \tau_3 < \infty. \tag{33}$$

where τ_3 has been introduced in (30). Since $e_2(\tau) = q_{2t} - q_2(\tau) \to 0$ as $\tau \to \tau_2^*$ where $\tau_2^* \leq \tau_2 + \tau_3$, the convergence result (33) leads to

$$(\tan q_{1t} - \tan q_{1a}(\tau)) \to 0 \quad \text{as} \quad \tau \to \tau_2^*, \qquad \tau_2^* \leq \tau_2 + \tau_3 < \infty. \tag{34}$$

Because in the FT case $e_a(\tau) = q_{1a}(\tau) - q_1(\tau)$ tends to zero as $\tau \to T_a$, $T_a < \infty$ (compare {41}), then finally (34) can be replaced with:

$$(\tan q_{1t} - \tan q_1(\tau)) \to 0 \quad \text{as} \quad \tau \to \tau_f, \qquad \tau_f < \tau_2^* + T_a < \infty. \tag{35}$$

Considerations related to formulas (20) to (23) are valid also in FT case. Thus according to (35) one can conclude that for the FT case holds:

$$e_1(\tau) \to 0 \quad \text{as} \quad \tau \to \tau_f, \qquad \tau_f < \tau_2^* + T_a < \infty. \tag{36}$$

Note that time instant τ_f has been estimated conservatively in (36).

3 Appendix

3.1 Derivation of relation (8)

According to (4) we have:

$$\boldsymbol{e}^* = \boldsymbol{L}^{-1}\boldsymbol{e}_L^* = \frac{1}{\|\boldsymbol{g}_{2t}^*\|^2}\begin{bmatrix} \sin q_{1t} & -\cos q_{1t}\cos q_{2t} \\ \cos q_{1t}\cos q_{2t} & \sin q_{1t}\end{bmatrix}\begin{bmatrix} e_{2L} \\ e_{3L}\end{bmatrix} = \begin{bmatrix} \frac{1}{\|\boldsymbol{g}_{2t}^*\|^2}(e_{2L}\sin q_{1t} - e_{3L}\cos q_{1t}\cos q_{2t}) \\ \frac{1}{\|\boldsymbol{g}_{2t}^*\|^2}(e_{2L}\cos q_{1t}\cos q_{2t} + e_{3L}\sin q_{1t})\end{bmatrix}. \tag{37}$$

Hence, we can write:

$$\|\boldsymbol{e}^*\|^2 = \frac{1}{\|\boldsymbol{g}_{2t}^*\|^4}\big(e_{2L}^2\sin^2 q_{1t} - 2e_{2L}e_{3L}\sin q_{1t}\cos q_{1t}\cos q_{2t} + e_{3L}^2\cos^2 q_{1t}\cos^2 q_{2t} + \\ + e_{2L}^2\cos^2 q_{1t}\cos^2 q_{2t} + 2e_{2L}e_{3L}\sin q_{1t}\cos q_{1t}\cos q_{2t} + e_{3L}^2\sin^2 q_{1t}\big)$$

and after reducing particular terms one gets

$$\|\boldsymbol{e}^*\| = \sqrt{\frac{1}{\|\boldsymbol{g}_{2t}^*\|^4}(e_{2L}^2\|\boldsymbol{g}_{2t}^*\|^2 + e_{3L}^2\|\boldsymbol{g}_{2t}^*\|^2)} = \sqrt{\frac{\|\boldsymbol{e}_L^*\|^2}{\|\boldsymbol{g}_{2t}^*\|^2}} = \frac{\|\boldsymbol{e}_L^*\|}{\|\boldsymbol{g}_{2t}^*\|}.$$

References

[1] M. Michałek and K. Kozłowski. The VFO state-constrained stabilisation of the nonholonomic manipulator with limited control input. *International Journal of Control*, 84(10):1678–1694, 2011.

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