

# Modular tracking controller for N-trailers with non-zero hitching offsets

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**Abstract**—The paper presents the concept and stability analysis of a trajectory-tracking feedback control system for truly N-trailer robots equipped with arbitrary number of trailers interconnected by the non-zero hitching offsets. Thanks to application of the cascaded-like control structure the considered solution is modular and highly scalable with respect to a number of trailers. Formal analysis of the closed-loop system provides sufficient conditions for asymptotic tracking of a whole set of the so-called *segment-platooning* reference trajectories with constant as well as time-varying curvature under assumption of sign-homogeneous hitching of trailers. Generality of the concept description together with formal analysis presented in the paper fills in some extent the gap which arose in the literature in the context of trajectory tracking with N-trailers of differentially non-flat kinematics. Simulation examples validate utility and modularity of the control method.

## I. INTRODUCTION

We are interested in the trajectory-tracking problem for truly N-trailer robotic vehicles (in short: N-trailers) comprising the unicycle-like tractor and *arbitrary number* of off-axle hitched trailers attached in a chain by the passive rotary joints (see Fig. 1). The N-trailers are practically important and very interesting hard-to-control systems due to specific properties of their kinematics [10], [9], [1], [7]. Most control solutions for truly N-trailers available in the literature concern time-non-critical tasks, i.e. the set-point stabilization and path-following problem, see e.g. [11], [2], [20], [3], [8] and [16]. Although numerous specialized tracking control laws were devised for tractor-trailer robots with strictly limited number of trailers (typically for  $N \leq 2$ ), only very limited number of tracking controllers have been proposed for truly N-trailers. In this context, one may recall for instance solutions proposed in [15], [6], [17] for the differentially flat Standard N-Trailer (SNT) robots equipped solely with on-axle hitches (for a notion of differential flatness the reader is referred to [19]). To the author's best knowledge, the only trajectory tracking control approach elaborated for differentially non-flat truly N-trailers equipped solely with off-axle hitches – the so-called non-Standard N-Trailers (nSNT) [7] – have been proposed in [4] by exploiting a cascaded-like control structure. However, the authors in [4] limited their considerations (among other restrictions imposed) only to backward tracking of constant-curvature reference trajectories (rectilinear and circular ones) assuming omnidirectional kinematics of a tractor. Furthermore, formal stability analysis of the closed-loop system provided in [4] was limited only to the case of a tractor-single-trailer system

( $N = 1$ ). Therefore, one may have a strong feeling that the control approach presented in [4] has not been fully explored to reveal its actual potential.

The aim of the current paper is to provide generic description and stability analysis of the modular trajectory tracking control system for truly nSNT robots based on the cascaded-like control concept presented in [4] (see also [13]). The new description substantially relaxes restrictions imposed in [4]. In particular, by introducing a set of *segment-platooning* reference trajectories (admitting both constant-curvature and varying-curvature trajectories) the sufficient conditions for local asymptotic stability of the closed-loop system are derived for both backward and forward motion strategies of a truly N-trailer vehicle with off-axle hitching, preserving location of the guidance point on a last trailer. Modularity of the concept is illustrated by exemplary simulations applying two alternative unicycle controllers in the outer loop of the cascaded-like control system.

## II. KINEMATICS OF N-TRAILERS AND ASSUMPTIONS

Configuration of the N-trailer can be uniquely determined by a vector (see Fig. 1 for geometrical interpretation)

$$\mathbf{q} \triangleq [\beta_1 \dots \beta_N \theta_N x_N y_N]^\top = \begin{bmatrix} \beta \\ \mathbf{q}_N \end{bmatrix} \in \mathbb{T}^N \times \mathbb{R}^3, \quad (1)$$

where  $\beta$  is the joint-angle vector, while  $\mathbf{q}_N$  denotes a posture of a last trailer (called the *guidance segment*) comprising the trailer-body orientation  $\theta_N$  and position coordinates  $x_N, y_N$  of the *guidance point*  $P$ . The control input to the vehicle is a velocity vector  $\mathbf{u}_0 = [\omega_0 v_0]^\top \in \mathbb{R}^2$  where  $\omega_0$  and  $v_0$  are the angular and longitudinal velocities of the unicycle-like tractor, respectively.

Kinematic structure of the N-trailer is characterized by two kinds of parameters (see Fig. 1): trailer lengths  $L_i > 0$  and hitching offsets  $L_{hi} \in \mathbb{R}$ ,  $i = 1, \dots, N$ . We adopt the sign convention where  $L_{hi} > 0$  if the  $i$ th joint is located *behind* the wheels-axle of the  $(i-1)$ st segment, and  $L_{hi} < 0$  in the opposite case. From now on we restrict considerations to the N-trailers which satisfy the following assumptions:

- A1.  $\forall i \in \{1, \dots, N\} L_{hi} \neq 0$ ,
- A2.  $L_{hi}L_{hj} > 0$  for all  $i, j \in \{1, \dots, N\}$ ,
- A3.  $\forall i \in \{1, \dots, N\} |L_{hi}| < L_i$  if  $L_{hi} < 0$ .

Assumption A1 confines considered N-trailers to the nSNT type. A2 assumes common signs for all the hitching offsets (all positive or all negative – *sign-homogeneous hitching*); necessity of introducing A2 is dictated by the need of the jackknife phenomenon avoidance in the proposed control system as it will be clarified in Section IV. For the first sight, A2 may seem substantially limiting, however most practical

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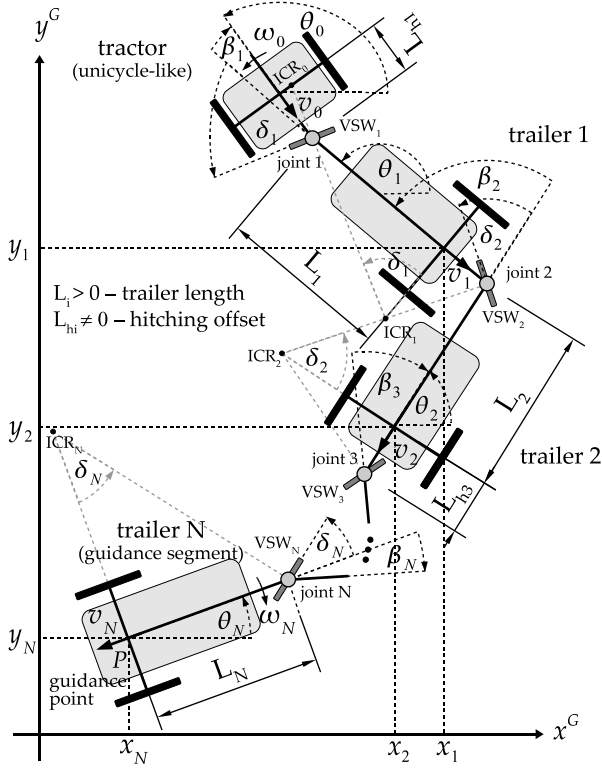


Fig. 1. Kinematic structure of the nSNT in a global frame  $\{x^G, y^G\}$ ;  $ICR_i$  and  $VSW_i$  denote, respectively, instantaneous center of rotation of the  $i$ th vehicle segment and virtual steering wheel of the  $i$ th joint (cf. [1]).

structures of N-trailer robots in fact meet this assumption. Limitation A3 is clearly justified by practical mechanical constraints.

Treating particular vehicle segments as unicycles

$$\dot{\theta}_i = \omega_i, \quad \dot{x}_i = v_i c\theta_i, \quad \dot{y}_i = v_i s\theta_i, \quad i = 0, \dots, N, \quad (2)$$

where  $\mathbf{u}_i = [\omega_i \ v_i]^\top \in \mathbb{R}^2$  is a velocity vector of the  $i$ th segment, one can find that the following geometrical relation holds for any two neighboring segments [7]:

$$\mathbf{u}_i = \begin{bmatrix} -\frac{L_{hi}}{L_i} c\beta_i & \frac{1}{L_i} s\beta_i \\ L_{hi} s\beta_i & c\beta_i \end{bmatrix} \mathbf{u}_{i-1} =: \mathbf{J}_i(\beta_i) \mathbf{u}_{i-1}, \quad (3)$$

where transformation matrix  $\mathbf{J}_i(\beta_i)$  is invertible for any  $\beta_i$  under assumption A1, leading to the inverse relation

$$\mathbf{u}_{i-1} = \begin{bmatrix} -\frac{L_i}{L_{hi}} c\beta_i & \frac{1}{L_{hi}} s\beta_i \\ L_i s\beta_i & c\beta_i \end{bmatrix} \mathbf{u}_i =: \mathbf{J}_i^{-1}(\beta_i) \mathbf{u}_i. \quad (4)$$

Following works [7] and [12] one may recall that kinematics of the N-trailers can be expressed as a drift-free system

$$\dot{\mathbf{q}} = \begin{bmatrix} \dot{\beta} \\ \dot{\mathbf{q}}_N \end{bmatrix} = \begin{bmatrix} \mathbf{S}_\beta(\beta) \\ \mathbf{S}_N(\beta, \mathbf{q}_N) \end{bmatrix} \mathbf{u}_0, \quad (5)$$

with sub-matrices

$$\mathbf{S}_\beta = \begin{bmatrix} \mathbf{c}^\top \Gamma_1(\beta_1) \\ \mathbf{c}^\top \Gamma_2(\beta_2) \mathbf{J}_1(\beta_1) \\ \vdots \\ \mathbf{c}^\top \Gamma_N(\beta_N) \mathbf{J}_{N-1}^1(\beta) \end{bmatrix}, \quad \mathbf{S}_N = \begin{bmatrix} \mathbf{d}^\top \mathbf{J}_N^1(\beta) \\ \mathbf{d}^\top \mathbf{J}_N^1(\beta) c\theta_N \\ \mathbf{d}^\top \mathbf{J}_N^1(\beta) s\theta_N \end{bmatrix}, \quad (6)$$

where  $\Gamma_i(\beta_i) \triangleq \mathbf{I} - \mathbf{J}_i(\beta_i)$ ,  $\mathbf{I} \in \mathbb{R}^{2 \times 2}$  is a unit matrix,  $\mathbf{J}_i^1(\beta) \triangleq \mathbf{J}_i(\beta_i) \dots \mathbf{J}_1(\beta_1)$ , and  $\mathbf{c}^\top \triangleq [1 \ 0]$ ,  $\mathbf{d}^\top \triangleq [0 \ 1]$ .

### III. CONTROL PROBLEM AND S-P TRAJECTORIES

Let us complement kinematics (5) with definition of the system output  $\mathbf{y} \triangleq \mathbf{q}_N = \mathbf{C}\mathbf{q}$ ,  $\mathbf{C} = [\mathbf{0}_{3 \times N} \ \mathbf{I}_{3 \times 3}]$ , being the posture of the guidance segment. Assume the *admissible* output reference trajectory is given

$$\mathbf{y}_r(t) = \mathbf{q}_{Nr}(t) = [\theta_{Nr}(t) \ x_{Nr}(t) \ y_{Nr}(t)]^\top \in \mathbb{R}^3 \quad (7)$$

satisfying unicycle kinematics (2), that is

$$\dot{\mathbf{q}}_{Nr} = \mathbf{G}(\mathbf{q}_{Nr}) \mathbf{u}_{Nr}, \quad \mathbf{G} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta_{Nr} & s\theta_{Nr} \end{bmatrix}^\top, \quad (8)$$

where  $\mathbf{u}_{Nr} = [\omega_{Nr} \ v_{Nr}]^\top \in \mathbb{R}^2$  is a reference velocity along  $\mathbf{y}_r(t)$  such that  $\omega_{Nr}(t), v_{Nr}(t), \dot{\omega}_{Nr}(t), \dot{v}_{Nr}(t) \in \mathcal{L}_\infty$ , and additionally  $\forall t \geq 0 \ \|\mathbf{u}_{Nr}(t)\| \neq 0$ . The latter requirement reflects a general *persistent excitation* (PE) condition for trajectory (7). Further, assume that the existence of (7) determines the existence of a bounded joint-angles reference trajectory

$$\beta_r(t) = [\beta_{1r}(t) \ \dots \ \beta_{Nr}(t)]^\top \in \mathbb{T}^N \quad (9)$$

which, according to (5), has to satisfy an equation of the exogenous system:

$$\dot{\beta}_r = \mathbf{S}_\beta(\beta_r) \mathbf{u}_{0r} \stackrel{(4)}{=} \mathbf{S}_\beta(\beta_r) \prod_{j=1}^N \mathbf{J}_j^{-1}(\beta_{jr}) \mathbf{u}_{Nr}. \quad (10)$$

According to the distinction proposed above, reference trajectory (7) determines desired behavior for the system output (posture of the guidance segment), whereas (9) defines a corresponding desired evolution for the *inner configuration* of a vehicle. Introducing the joint-angles error and posture error, respectively, as

$$\tilde{\beta} = \begin{bmatrix} \tilde{\beta}_1 \\ \vdots \\ \tilde{\beta}_N \end{bmatrix} \triangleq \beta_r - \beta, \quad \mathbf{e}_N = \begin{bmatrix} e_\theta \\ e_x \\ e_y \end{bmatrix} \triangleq \begin{bmatrix} \theta_{Nr} - \theta_N \\ x_{Nr} - x_N \\ y_{Nr} - y_N \end{bmatrix} \quad (11)$$

one can formulate a trajectory-tracking control (TTC) problem as follows.

*Definition 1 (TTC Problem):* For nSNT kinematics (5), satisfying assumptions A1-A3, find a feedback control input  $\mathbf{u}_0 = \mathbf{u}_0(\beta, \mathbf{e}_N, \cdot)$  which guarantees asymptotic tracking in the sense that  $\tilde{\beta}(t) \rightarrow \mathbf{0}$  and  $\mathbf{e}_N(t) \rightarrow \mathbf{0}_{2\mu\pi}$  as  $t \rightarrow \infty$ , where  $\mathbf{0}_{2\mu\pi} \triangleq [2\mu\pi \ 0 \ 0]^\top$ ,  $\mu \in \{0, \pm 1, \pm 2, \dots\}$ .

*Remark 1:* Introduction of the *zero-set*  $\mathbf{0}_{2\mu\pi}$  in the above definition (which includes  $\mathbf{0}$  as a special case for  $\mu = 0$ ) has been motivated by a nature of the angular error  $e_\theta$ . Using the zero-set in definition of TTC Problem will enable considering a wider class of control laws which can solve it. For simplicity, we will write  $\mathbf{e}_N = \mathbf{0}_{2\mu\pi}$  to indicate that  $\mathbf{e}_N$  is an element of zero-set for some value of  $\mu$ .

Solution to the above stated TTC Problem will be given in the next section for the set of so-called *segment-platooning* (S-P) reference trajectories. The S-P reference signals have

been introduced for the first time in [12] in the context of the path-following task. By analogy, we will call the reference trajectory  $\mathbf{q}_r(t) = [\boldsymbol{\beta}_r^\top(t) \mathbf{q}_{N_r}^\top(t)]^\top$  as S-P if it ensures satisfaction of the following relation

$$\forall t \geq 0 \quad v_{i-1r}(t) \cdot v_{ir}(t) > 0, \quad i = 1, \dots, N, \quad (12)$$

which means that reference longitudinal velocities of any two neighboring segments are non-zero and have the same signs along a reference trajectory defined by (7)-(8) and (9)-(10). It is easy to find that  $\mathbf{q}_r(t) = [\mathbf{0} \ \mathbf{q}_{N_r}^\top(t)]^\top$  corresponding to rectilinear output trajectory  $\mathbf{q}_{N_r}(t)$  belongs to set S-P. Further, it can be shown that  $\mathbf{q}_r(t) = [\boldsymbol{\beta}_r^\top \ \mathbf{q}_{N_r}^\top(t)]^\top$  corresponding to circular output trajectory  $\mathbf{q}_{N_r}(t)$  is S-P when  $\forall i \in \{1, \dots, N\} \ \beta_{ir} \in (-\gamma_i, \gamma_i)$ , where<sup>1</sup>

$$\gamma_i = \arccos\left(-\min\left\{\frac{L_{hi}}{L_i}, \frac{L_i}{L_{hi}}\right\}\right) \quad \text{if } L_{hi} > 0, \quad (13)$$

$$\gamma_i = \arccos(|L_{hi}|/L_i) \quad \text{if } L_{hi} < 0. \quad (14)$$

For more general reference trajectories  $\mathbf{q}_r(t)$ , corresponding to varying-curvature output trajectories  $\mathbf{q}_{N_r}(t)$ , satisfaction of (12) can be numerically checked off-line by using formula

$$\begin{bmatrix} \omega_{ir} \\ v_{ir} \end{bmatrix} \stackrel{(4)}{=} \prod_{j=i+1}^N \mathbf{J}_j^{-1}(\beta_{jr}) \mathbf{u}_{N_r}, \quad i = 0, \dots, N-1, \quad (15)$$

which holds along any  $\mathbf{q}_r(t)$  with reference joint-angles (9) being a 'steady-state' solution of equation (10).

#### IV. MODULAR TRACKING CONTROLLER

##### A. Cascaded-like control structure for nSNT kinematics

The concept of a cascaded-like controller comes from inverse relation (4), which upon A1 allows one to write

$$\mathbf{u}_0 = \prod_{j=1}^N \mathbf{J}_j^{-1}(\beta_j) \mathbf{u}_N. \quad (16)$$

The above equation reflects how velocity  $\mathbf{u}_N$  of the guidance segment can be forced by tractor input  $\mathbf{u}_0$  in the nSNT kinematics. Because (16) is a purely algebraic mapping, it is possible to directly influence motion of the last trailer by the tractor control input. Thus, the natural choice for the tractor control-input results from equation

$$\mathbf{u}_0(\boldsymbol{\beta}, \boldsymbol{\Phi}) = \begin{bmatrix} \omega_0(\boldsymbol{\beta}, \boldsymbol{\Phi}) \\ v_0(\boldsymbol{\beta}, \boldsymbol{\Phi}) \end{bmatrix} \triangleq \prod_{j=1}^N \mathbf{J}_j^{-1}(\beta_j) \boldsymbol{\Phi}(e_N, t), \quad (17)$$

where  $\boldsymbol{\Phi} = [\Phi_\omega \ \Phi_v]^\top \in \mathbb{R}^2$  is some feedback control function such that its direct application into unicycle kinematics of the guidance segment (i.e. by taking  $\mathbf{u}_N := \boldsymbol{\Phi}(e_N, t)$ ) would guarantee asymptotic tracking of the output reference trajectory  $\mathbf{q}_{N_r}(t)$ . Hence,  $\boldsymbol{\Phi}(e_N, t)$  represents one of the tracking control laws available in the literature and devised to the unicycle (see e.g. [14], [5]). To keep further considerations general enough, assume only that  $\boldsymbol{\Phi}(e_N(t), t)$  has some desired properties, namely:

P1.  $\forall t \geq 0 \ \|\boldsymbol{\Phi}(e_N(t), t)\| < \infty$ ,

<sup>1</sup>Conditions (13)-(14) are less conservative than those proposed in [12].

P2.  $\mathbf{u}_N(t) = \boldsymbol{\Phi}(t) \Rightarrow \|\mathbf{e}_N(t)\| < \infty \wedge \mathbf{e}_N(t) \xrightarrow{t \rightarrow \infty} \mathbf{0}_{2\mu\pi}$ ,

P3.  $\boldsymbol{\Phi}(\mathbf{0}_{2\mu\pi}, t) \equiv \mathbf{u}_{N_r}(t)$ .

Property P1 claims boundedness of control function  $\boldsymbol{\Phi}$ . P2 means that making velocity  $\mathbf{u}_N(t)$  of the guidance segment modeled by (2) equal to the control function  $\boldsymbol{\Phi}(t)$  leads to the bounded and asymptotically convergent posture error  $e_N$  (i.e. P2 reflects efficiency of  $\boldsymbol{\Phi}$  when it is directly applied into unicycle kinematics). P3 indicates that  $\boldsymbol{\Phi}$  is well determined also along the reference trajectory (i.e. for  $e_N = \mathbf{0}_{2\mu\pi}$ ).

Worth noting that (17) defines in fact a cascade interconnection of the outer-loop tracking controller (dedicated to the guidance segment) represented by  $\boldsymbol{\Phi}(e_N, t)$ , and the inner-loop velocity transformation (ILT) being the product of matrices  $\mathbf{J}_j^{-1}(\beta_j)$  for  $j = 1, \dots, N$  evaluated at current angular configuration of the vehicle chain. A general scheme explaining the considered control structure is shown in Fig. 2.

*Proposition 1:* Cascaded-like control law (17) with feedback control function  $\boldsymbol{\Phi}(e_N(t), t)$  possessing properties P1-P3 solves the TTC Problem for the S-P reference trajectories  $\mathbf{q}_r(t) = [\boldsymbol{\beta}_r^\top(t) \mathbf{q}_{N_r}^\top(t)]^\top$  at least locally in the neighborhood of point  $(\boldsymbol{\beta}, e_N) = (\mathbf{0}, \mathbf{0}_{2\mu\pi})$  under the following conditions:

- C1.  $\forall t \geq 0 \ \text{sgn}(v_{Nr}(t)) = -\text{sgn}(L_{hi}), i = 1, \dots, N$ ,
- C2.  $\forall t \geq 0 \ \|\mathbf{u}_{N_r}(t)\| \leq \delta_1$  and  $\|\dot{\mathbf{u}}_{N_r}(t)\| \leq \delta_2$  for sufficiently small constants  $\delta_1, \delta_2 > 0$  in the case of reference trajectories for which  $\dot{\mathbf{u}}_{N_r}(t) \neq \mathbf{0}$ .

Condition C1 restricts a sign of the longitudinal reference velocity for the guidance segment by permitting only the backward tracking strategy if all the hitching offsets are positive, or forward tracking if the hitching offsets are negative. C1 justifies introduction of the sign-homogeneity assumption A2 formulated in Section II. Condition C2 claims sufficiently slow and smooth reference motion along  $\mathbf{q}_{N_r}(t)$ , although it concerns only trajectories with time-varying reference velocities.

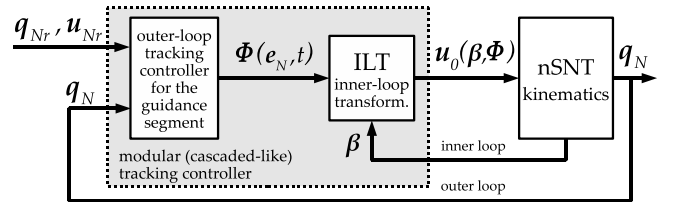


Fig. 2. Block scheme of the cascaded-like tracking controller for nSNT.

##### B. Proof of Proposition 1

One may easily show boundedness of control vector (17), which directly results from property P1 and from boundedness of the Frobenius norm of matrix  $\mathbf{J}_j^{-1}(\beta_j)$  under assumption A1.

Let us examine closed-loop behavior of the guidance segment. Upon (3) and (17) one can write

$$\mathbf{u}_N = \prod_{j=N}^1 \mathbf{J}_j(\beta_j) \prod_{j=1}^N \mathbf{J}_j^{-1}(\beta_j) \boldsymbol{\Phi}(e_N, t) = \boldsymbol{\Phi}(e_N, t).$$

Thus, one concludes that application of control law (17) makes the guidance segment move in a way as it would

be directly controlled by function  $\Phi(e_N, t)$ . Now, according to property P2, one ascertains boundedness of posture error  $e_N(t)$  and its asymptotic convergence to  $\mathbf{0}_{2\mu\pi}$ . The above conclusion is valid for both constant-curvature as well as varying-curvature reference trajectories regardless they are of the S-P type or not.

Now, we shall consider stability of the joint-angles error dynamics. Define the outer-loop control difference

$$\tilde{\Phi}(t) \triangleq \mathbf{u}_{Nr}(t) - \Phi(e_N(t), t). \quad (18)$$

By taking a time-derivative of error  $\tilde{\beta}$  and utilizing (5), (10), and (18) one obtains the joint-angles error dynamics:

$$\begin{aligned} \dot{\tilde{\beta}} &= \mathbf{S}_\beta(\beta_r) \prod_{j=1}^N \mathbf{J}_j^{-1}(\beta_{jr}) \mathbf{u}_{Nr} \\ &\quad - \mathbf{S}_\beta(\beta_r - \tilde{\beta}) \prod_{j=1}^N \mathbf{J}_j^{-1}(\beta_{jr} - \tilde{\beta}_j) (\mathbf{u}_{Nr} - \tilde{\Phi}). \end{aligned} \quad (19)$$

One may easily check that  $(\tilde{\beta} = \mathbf{0}, \tilde{\Phi} = \mathbf{0})$  is an equilibrium of dynamics (19). Closer investigation of equation (19) (see the form of matrix  $\mathbf{S}_\beta$  in (6)) reveals its upper-triangular form, where the  $i$ th row can be written (after some algebraic manipulations) as

$$\dot{\tilde{\beta}}_i = f_i(\tilde{\beta}_i^N, \beta_r, \mathbf{u}_{Nr}) + g_i(\tilde{\beta}_i^N, \beta_r, \tilde{\Phi}), \quad i = 1, \dots, N,$$

where  $\tilde{\beta}_i^N \triangleq [\tilde{\beta}_i \ \tilde{\beta}_{i+1} \ \dots \ \tilde{\beta}_N]^\top$ , and

$$\begin{aligned} f_i &= \mathbf{c}^\top [\mathbf{I} - \mathbf{J}_i(\beta_{ir})] \prod_{j=i}^N \mathbf{J}_j^{-1}(\beta_{jr}) \mathbf{u}_{Nr} \\ &\quad - \mathbf{c}^\top [\mathbf{I} - \mathbf{J}_i(\beta_{ir}) \mathbf{R}_{hi}(\tilde{\beta}_i)] \prod_{j=i}^N \mathbf{J}_j^{-1}(\beta_{jr}) \mathbf{R}_j(\tilde{\beta}_j) \mathbf{u}_{Nr}, \\ g_i &= \mathbf{c}^\top [\mathbf{I} - \mathbf{J}_i(\beta_{ir}) \mathbf{R}_{hi}(\tilde{\beta}_i)] \prod_{j=i}^N \mathbf{J}_j^{-1}(\beta_{jr}) \mathbf{R}_j(\tilde{\beta}_j) \tilde{\Phi}, \end{aligned}$$

with matrices

$$\mathbf{R}_{hi}(\tilde{\beta}_i) \triangleq \begin{bmatrix} \tilde{\beta}_i & \frac{s\tilde{\beta}_i}{L_{hi}} \\ -L_{hi}s\tilde{\beta}_i & \tilde{\beta}_i \end{bmatrix}, \quad \mathbf{R}_i(\tilde{\beta}_i) \triangleq \begin{bmatrix} \tilde{\beta}_i & \frac{s\tilde{\beta}_i}{L_i} \\ -L_i s\tilde{\beta}_i & \tilde{\beta}_i \end{bmatrix}.$$

Treating  $\tilde{\beta}$  as a state and  $\tilde{\Phi}$  as an input, one can linearize (19) around equilibrium  $(\tilde{\beta} = \mathbf{0}, \tilde{\Phi} = \mathbf{0})$  to yield the approximate dynamics

$$\dot{\tilde{\beta}} = \mathbf{A}(\beta_r, \mathbf{u}_{Nr}) \tilde{\beta} + \mathbf{B}(\beta_r) \tilde{\Phi}, \quad (20)$$

where

$$\mathbf{A}(\beta_r, \mathbf{u}_{Nr}) = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ 0 & a_{22} & \dots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{NN} \end{bmatrix}, \quad \mathbf{B}(\beta_r) = \begin{bmatrix} \mathbf{b}_1^\top \\ \mathbf{b}_2^\top \\ \vdots \\ \mathbf{b}_N^\top \end{bmatrix}$$

with diagonal elements

$$a_{ii} = \left[ \frac{L_i s \beta_{ir}}{L_{hi}} \quad \frac{c \beta_{ir}}{L_{hi}} \right] \prod_{j=i+1}^N \mathbf{J}_j^{-1}(\beta_{jr}) \mathbf{u}_{Nr}, \quad 1 \leq i < N, \quad (21)$$

$$a_{NN} = \left[ \frac{L_N s \beta_{Nr}}{L_{hN}} \quad \frac{c \beta_{Nr}}{L_{hN}} \right] \mathbf{u}_{Nr}, \quad (22)$$

with non-zero off-diagonal elements

$$\begin{aligned} a_{il} &= \left[ \left(1 + \frac{L_i c \beta_{ir}}{L_{hi}}\right) \quad \frac{-s \beta_{ir}}{L_{hi}} \right] \prod_{j=i+1}^l \mathbf{J}_j^{-1}(\beta_{jr}) \\ &\quad \times \begin{bmatrix} 0 & \frac{1}{L_l} \\ -L_l & 0 \end{bmatrix} \prod_{j=l+1}^N \mathbf{J}_j^{-1}(\beta_{jr}) \mathbf{u}_{Nr} \end{aligned} \quad (23)$$

valid for  $i = 1, \dots, N-1$ ,  $l = i+1, \dots, N$ , and with the  $i$ th row of matrix  $\mathbf{B}(\beta_r)$  of the form:

$$\mathbf{b}_i^\top = \left[ \left(1 + \frac{L_i c \beta_{ir}}{L_i}\right) \quad \frac{-s \beta_{ir}}{L_i} \right] \prod_{j=i}^N \mathbf{J}_j^{-1}(\beta_{jr}), \quad i = 1, \dots, N.$$

Based on the previous considerations, we know that posture error  $e_N(t)$  converges to  $\mathbf{0}_{2\mu\pi}$  independently of joint-angles error  $\tilde{\beta}(t)$ . In this context, (20) can be understood as (approximate) *inner dynamics* of the closed-loop system. Upon property P3 and definition (18) it is clear that  $\tilde{\Phi} = \mathbf{0}$  for  $e_N = \mathbf{0}_{2\mu\pi}$ . Thus, one can treat  $\tilde{\Phi}$  in (20) as a bounded perturbation vanishing in time. As a consequence, one may analyze stability of dynamics (20) under perfect output-tracking conditions, that is for  $e_N = \mathbf{0}_{2\mu\pi}$  and  $\tilde{\Phi} = \mathbf{0}$ . In this case linear dynamics

$$\dot{\tilde{\beta}} = \mathbf{A}(\beta_r, \mathbf{u}_{Nr}) \tilde{\beta} \quad (24)$$

describes the approximate *zero-dynamics* of the closed-loop system. Let us analyze properties of state-matrix  $\mathbf{A}(\beta_r, \mathbf{u}_{Nr})$ . Since  $\mathbf{A}$  has the upper-triangular form, the eigenvalues  $\lambda_i(\mathbf{A})$ ,  $i = 1, \dots, N$  correspond to its diagonal elements. According to (21)-(22) and recalling (15) and (4) one can write  $a_{ii} = v_{i-1r}/L_{hi} = \text{sgn}(v_{i-1r}) |v_{i-1r}|/L_{hi}$  for  $i = 1, \dots, N$ . Now, for the S-P reference trajectories we can utilize (12) which, under condition C1, yields for  $i = 1, \dots, N$

$$a_{ii} = \frac{\text{sgn}(v_{Nr}) |v_{i-1r}|}{L_{hi}} = \frac{-|v_{i-1r}|}{|L_{hi}|} \leq -\alpha, \quad (25)$$

where the bound

$$\alpha = \min_{i \in \{1, \dots, N\}} \left\{ \inf_{t \geq 0} \left| \frac{v_{i-1r}}{L_{hi}} \right| \right\} > 0 \quad (26)$$

is strictly positive for S-P trajectories thanks to the acute inequality in (12). As a consequence, one may state that for  $t \geq 0$  all the eigenvalues of matrix  $\mathbf{A}$  are real-negative for the S-P reference trajectories (regardless  $\mathbf{u}_{Nr}$  is constant or time-varying). Now, we must separately consider two possible cases: when reference velocity  $\mathbf{u}_{Nr}$  is constant, and when it is time-varying. It is well known that in the former case the reference joint-angles  $\beta_r$  are constant too, and the state-matrix  $\mathbf{A}(\beta_r, \mathbf{u}_{Nr})$  becomes time-invariant. As a consequence, exponential stability of (24) at  $\tilde{\beta} = \mathbf{0}$  results directly from (25)-(26) in this case. If velocity  $\mathbf{u}_{Nr}$  is time-varying, one has to proceed investigation of properties of matrix  $\mathbf{A}(\beta_r(t), \mathbf{u}_{Nr}(t)) = \mathbf{A}(t)$  and its time-derivative  $\dot{\mathbf{A}}(t)$  to assess asymptotic stability of LTV system (24). To this aim, one finds that  $\|\mathbf{A}(\beta_r(t), \mathbf{u}_{Nr}(t))\| < \bar{A} < \infty \ \forall t \geq 0$  by taking into account the forms of

components (21)-(23) under assumption A1. Next, one can write  $\dot{a}_{ij}(\boldsymbol{\beta}_r, \mathbf{u}_{Nr}) = \mathbf{a}_{\beta_{ij}}^\top \dot{\boldsymbol{\beta}}_r + \mathbf{a}_{u_{ij}}^\top \dot{\mathbf{u}}_{Nr}$ ,  $i, j \in \{1, \dots, N\}$ , where  $\mathbf{a}_{\beta_{ij}}^\top \triangleq \partial a_{ij} / \partial \boldsymbol{\beta}_r$  and  $\mathbf{a}_{u_{ij}}^\top \triangleq \partial a_{ij} / \partial \mathbf{u}_{Nr}$ . According to (21)-(23), assumption A1, and assumed properties of reference velocity  $\mathbf{u}_{Nr}$  (see Section III) one may conclude

$$\|\mathbf{a}_{\beta_{ij}}^\top\| \leq \delta_{\beta_{ij}}, \quad \|\mathbf{a}_{u_{ij}}^\top\| \leq \delta_{u_{ij}}, \quad (27)$$

where  $\delta_{\beta_{ij}} > 0$  and  $\delta_{u_{ij}} > 0$  are some finite upper bounds. Furthermore, by recalling (10) and the form of matrix  $\mathbf{S}_\beta(\boldsymbol{\beta}_r)$  in (6) one may (conservatively) assess

$$\|\dot{\boldsymbol{\beta}}_r\| \leq \left\| \mathbf{S}_\beta(\boldsymbol{\beta}_r) \prod_{j=1}^N \mathbf{J}_j^{-1}(\beta_{jr}) \right\| \|\mathbf{u}_{Nr}\| \leq \delta_\beta \|\mathbf{u}_{Nr}\|,$$

where  $0 < \delta_\beta < \infty$  under assumption A1. Now, by using the above inequality together with (27), and condition C2 one can write  $\forall t \geq 0 \quad |\dot{a}_{ij}(\boldsymbol{\beta}_r(t), \mathbf{u}_{Nr}(t))| \leq \delta_{\beta_{ij}} \delta_\beta \delta_1 + \delta_{u_{ij}} \delta_2$ , and consequently (using a spectral norm of  $\mathbf{A}$ ):

$$\begin{aligned} \|\dot{\mathbf{A}}(\boldsymbol{\beta}_r(t), \mathbf{u}_{Nr}(t))\| &\leq N \max_{i,j} |\dot{a}_{ij}(\boldsymbol{\beta}_r(t), \mathbf{u}_{Nr}(t))| \\ &\leq N (\delta_{\beta_{ij}} \delta_\beta \delta_1 + \delta_{u_{ij}} \delta_2) \end{aligned} \quad (28)$$

for all  $t \geq 0$ , where  $\bar{\delta}_{\beta_{ij}} = \max_{i,j}(\delta_{\beta_{ij}})$  and  $\bar{\delta}_{u_{ij}} = \max_{i,j}(\delta_{u_{ij}})$ . By ensuring that  $\delta_1$  and  $\delta_2$  are sufficiently small (see C2) the right-hand side of (28) can be made small enough to satisfy the sufficient condition for asymptotic stability of LTV system (24), see [18]. Since (28) reflects the sufficient condition (not the necessary one), it may seem quite conservative from the practical viewpoint. It turns out however, that the closed-loop stability can be preserved in practice even for relatively large upper bounds  $\delta_1, \delta_2$ .  $\square$

*Remark 2:* Due to the lack of differential flatness of nSNT kinematics, it is not clear in general how to find  $\boldsymbol{\beta}_r(t)$  in other way than by numerical integration of (10). Note, however, that control law (17) does not utilize reference joint-angles (9) in any way, thus computation of  $\boldsymbol{\beta}_r(t)$  is not required for execution of the tracking task.

## V. SIMULATION RESULTS

Effectiveness and modularity of the cascaded-like tracking control law (17) are illustrated for nS3T kinematics by two simulation examples, SimS and SimV, corresponding to two alternative control functions, respectively, Samson's function  $\Phi^S$  and VFO function  $\Phi^V$ , used in the outer loop. Both control functions satisfy, at least locally, properties P1 to P3 under appropriate conditions – for details the reader is referred to [5] and [6]. Definitions of the control functions are provided below.

$$\Phi^S = \begin{bmatrix} \Phi_\omega^S \\ \Phi_v^S \end{bmatrix} \triangleq \begin{bmatrix} \omega_{Nr} + k_0 v_{Nr} \tilde{e}_3 s e_\theta / e_\theta + k_1(\mathbf{u}_{Nr}) e_\theta \\ v_{Nr} c e_\theta + k_2(\mathbf{u}_{Nr}) \tilde{e}_2 \end{bmatrix},$$

where  $\tilde{e}_2 = e_x c \theta_N + e_y s \theta_N$  and  $\tilde{e}_3 = -e_x s \theta_N + e_y c \theta_N$ ,  $k_0 > 0$  is a positive design parameter, while  $k_1(\mathbf{u}_{Nr}) = k_2(\mathbf{u}_{Nr}) \triangleq 2\xi \sqrt{\omega_{Nr}^2 + k_0 v_{Nr}^2}$  are the positive definite functions of reference velocity  $\mathbf{u}_{Nr}$  and coefficient  $\xi > 0$ .

$$\Phi^V = \begin{bmatrix} \Phi_\omega^V \\ \Phi_v^V \end{bmatrix} \triangleq \begin{bmatrix} k_a(\theta_{Na} - \theta_N) + \dot{\theta}_{Na} \\ h_x c \theta_N + h_y s \theta_N \end{bmatrix},$$

where  $h_x = k_p e_x + \dot{x}_{Nr}$ ,  $h_y = k_p e_y + \dot{y}_{Nr}$ ,  $k_a, k_p > 0$  are the design parameters, while

$$\theta_{Na} \triangleq \text{Atan2c}(v_{Nr} h_y, v_{Nr} h_x) \in \mathbb{R}, \quad (29)$$

is the auxiliary variable, the time-derivative of which takes the form  $\dot{\theta}_{Na} = (\dot{h}_y h_x - h_y \dot{h}_x) / (h_x^2 + h_y^2)$  for  $h_x^2 + h_y^2 \neq 0$ .

Simulation results obtained with control functions  $\Phi^S$  and  $\Phi^V$  are provided in Figs. 3 and 4, respectively. The admissible reference trajectory for the guidance segment has been a numerical solution of (8) for initial condition  $\mathbf{q}_{3r}(0) = [\frac{\pi}{2} \ -2 \ 0]^\top$  by taking reference velocities  $v_{3r} = \sigma \cdot 0.2 \text{ m/s}$ ,  $\omega_{3r} = (-0.15\sigma + 0.15 \sin 0.3t) \text{ rad/s}$  with  $\sigma = -1$  (backward motion strategy) for SimS, and  $\sigma = +1$  (forward motion strategy) for SimV. The following kinematic parameters and design coefficients have been used:  $L_i = 0.25 \text{ m}$ ,  $L_{hi} = +0.05 \text{ m}$  for SimS and  $L_{hi} = -0.05 \text{ m}$  for SimV,  $i = 1, 2, 3$ ,  $k_p = 1$ ,  $k_a = 2$ ,  $\xi = 1$ ,  $k_0 = 10$ . Initial configuration of the vehicle was set to  $\mathbf{q}(0) = [0 \ 0 \ 0 \ \frac{\pi}{2} \ -1.5 \ 0]^\top$  and has been highlighted in magenta in Figs. 3-4. To show convergence of joint-angle errors, reference signals (9) were computed by numerical integration of (10) for initial condition  $\boldsymbol{\beta}_r(0) = \mathbf{0}$ .

Analyzing the results one can find non-oscillatory motion character of the guidance segment obtained together with fast and agile transient maneuvers of the N-trailers, which ultimately lead to the asymptotic convergence of all the tracking errors to zero. Time-plots of reference longitudinal velocities  $v_{ir}$ ,  $i = 0, 1, 2$  clearly indicate that selected varying-curvature reference trajectories belong to the S-P set in both cases. Worth noting relatively low control cost for almost whole control time-horizon (initial peaks of  $\omega_0$  equal approximately to 106 and 295 rad/s, respectively, have not been shown for clarity of the plots).

## VI. CONCLUSIONS

The cascaded-like control law considered in the paper is a modular and highly scalable solution to the trajectory-tracking problem for nSNT kinematics. Modularity comes from the fact that not a particular one but a whole set of control functions satisfying P1-P3 can be used in the outer loop according to the needs of a designer. Scalability means that a change of a trailers number in a vehicle chain influences only a number of matrix multiplications required in the inner-loop transformation. To the author's best knowledge, the paper for the first time provides sufficient conditions of asymptotic tracking with differentially non-flat truly N-trailers for a wide set of S-P reference trajectories. Adaptation of the method to the GNT kinematics still remains an open problem.

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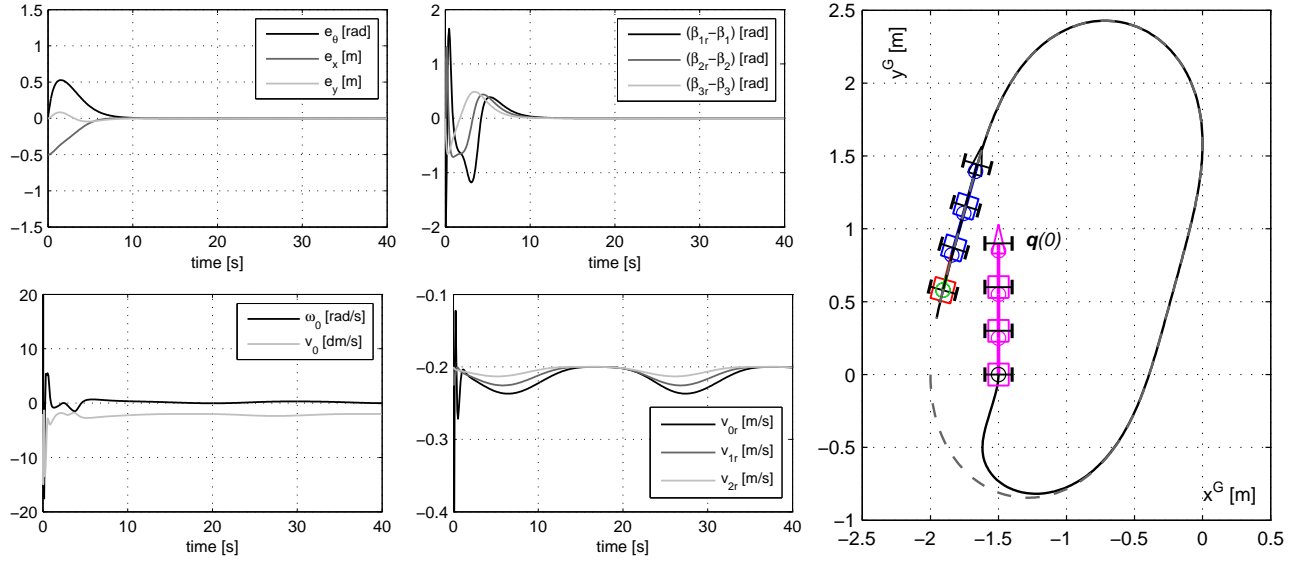


Fig. 3. SimS: Backward-tracking control performance obtained with control function  $\Phi^S$  used in the outer loop of controller (17)

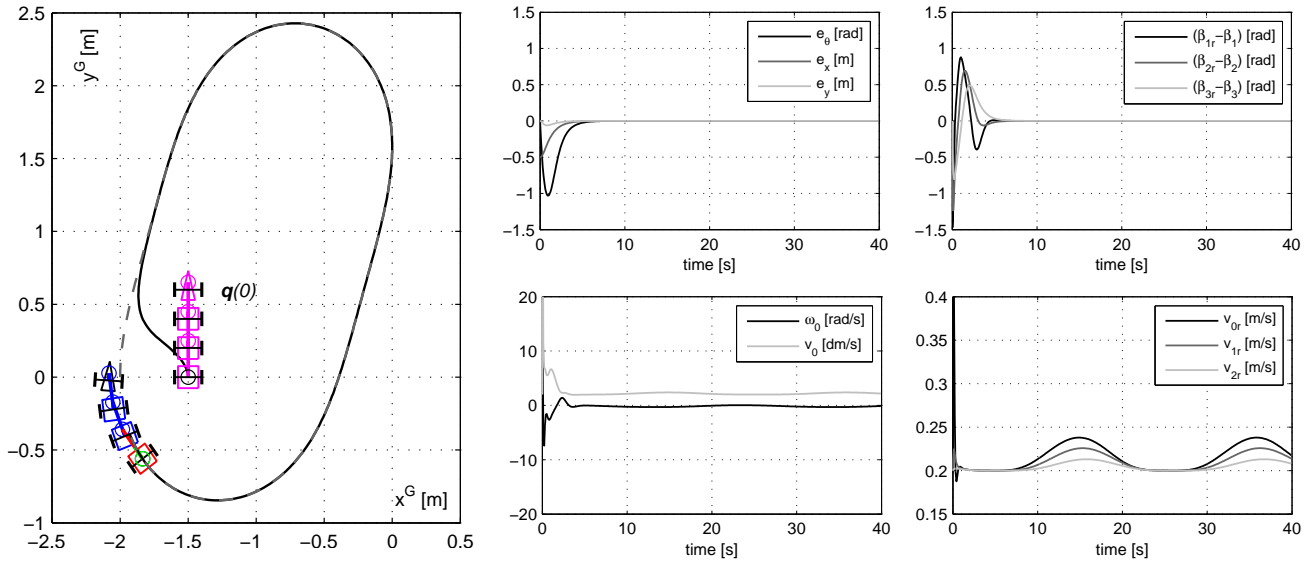


Fig. 4. SimV: Forward-tracking control performance obtained with control function  $\Phi^V$  used in the outer loop of controller (17)

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